hep-th/0505212
CPHT-RR 029.0505
SPhT-T05/079
LPTENS-05/16
SU-ITP-05/18

Generalized structures of $\mathcal{N}=1$ vacua

Mariana Graña[a,b], Ruben Minasian[c,a], Michela Petrini[a,d] and Alessandro Tomasiello[e]

[a] *Centre de Physique Théorique, Ecole Polytechnique*
91128 Palaiseau Cedex, France

[b] *Laboratoire de Physique Théorique, Ecole Normale Supérieure*
24, Rue Lhomond 75231 Paris Cedex 05, France

[c] *CEA/DSM/SPhT, Unité de recherche associée au CNRS, CEA/Saclay*
91191 Gif-sur-Yvette, France

[d] *Laboratoire de Mathématiques et Physique Théorique, Université François Rabelais*
Parc de Grandmont 37200, Tours, France

[e] *ITP, Stanford University, Stanford CA 94305-4060*

Abstract

We characterize $\mathcal{N}=1$ vacua of type II theories in terms of generalized complex structure on the internal manifold M. The structure group of $T(M)\oplus T^*(M)$ being SU(3)×SU(3) implies the existence of two pure spinors Φ_1 and Φ_2. The conditions for preserving $\mathcal{N}=1$ supersymmetry turn out to be simple generalizations of equations that have appeared in the context of $\mathcal{N}=2$ and topological strings. They are $(d+H\wedge)\Phi_1=0$ and $(d+H\wedge)\Phi_2=F_{\rm RR}$. The equation for the first pure spinor implies that the internal space is a twisted generalized Calabi-Yau manifold of a hybrid complex-symplectic type, while the RR-fields serve as an integrability defect for the second.

February 1, 2008

1 Introduction

In compactifications of type II theories to four dimensions, requiring unbroken $\mathcal{N}=2$ supersymmetry in the absence of fluxes leads to a simple and well–studied condition: the six–dimensional internal manifold should be Calabi–Yau. In recent years, there have been many attempts to find a similarly sleek answer in the presence of fluxes, both for $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetries in four dimensions.

Calabi-Yau manifolds admit a covariantly constant spinor. This condition has two parts, an algebraic one, namely the existence of a globally defined non-vanishing spinor; and a differential one, the requirement that it be covariantly constant. The first requirement cannot be relaxed if we want to have in the four–dimensional theory eight well–defined supercharges, or in other words an $\mathcal{N}=2$ effective action. Vacua with spontaneously broken supercharges or four–dimensional vacua other than Minkowski require relaxing the condition of the spinor to be covariantly constant.

Thus the existence of supersymmetric vacua of four–dimensional $\mathcal{N}=2$ effective theories in compactifications of type II theories in presence of NS and RR fluxes imposes a number of algebraic and differential conditions on the internal space. This paper is devoted to analysis of these conditions. The algebraic part says that the tangent plus cotangent bundle of the internal manifold, $T\oplus T^*$, must have structure group SU(3)×SU(3) [1–3] (see also [4] for the case of $G_2\times G_2$ structures). This is the generic condition ensuring $\mathcal{N}=2$ supersymmetry of the effective action in four-dimensions [5] and implies in particular the existence of two nowhere vanishing globally defined Clifford$(6,6)$ pure spinors Both can be represented just as sums of usual differential forms of even or odd rank over the manifold, and are denoted as Φ_1 and Φ_2. The differential part then says that the two pure spinors should satisfy

$$e^{-2A+\phi}(d+H\wedge)(e^{2A-\phi}\Phi_1)=0\,,\qquad e^{-2A+\phi}(d+H\wedge)(e^{2A-\phi}\Phi_2)=dA\wedge\bar{\Phi}_2+F\,.\quad(1.1)$$

Here, e^{2A} is the warp factor, ϕ the dilaton, H the NS three–form and F a weighted sum of the RR fields, see eqs.(3.1–3.4). Φ_1 has the same parity as the RR fields, namely even in IIA and odd in IIB. The equations above together with the equations for the norm of the pure spinors, (3.10), give the necessary and sufficient conditions on the manifold and the fluxes to have unbroken $\mathcal{N}=1$ supersymmetry on warped Minkowski space[1]. Equations (3.1–3.4) also take into account the possibility of a nonzero cosmological constant in four dimensions.

The normalizations of the pure spinors are such that these equations are not applicable (or rather, do not give any information, as the norms are zero in that case) to the case in which RR fields are absent. However the supersymmetry conditions for zero RR fluxes are very well known. As we will review later, the NS flux in that case does not enter in the equations as a simple twist of the exterior derivative, as it does in (1.1) (or in more complete form in (3.1–3.4)).

Besides being compact, these equations fit very well in the mathematical framework of generalized complex geometry [1,2]. In particular, the first equation in (1.1) implies that the manifold must be twisted generalized Calabi–Yau. This fact has

[1] In order to satisfy all the equations of motion, we have to impose additionally Bianchi identities and the equations of motion for the fluxes (see for example [6]).

been noted already in [7] for cases in which T has SU(3) structure. The SU(3) structure vacua are however just special cases of the more general SU(3)×SU(3) on $T \oplus T^*$ considered here. The SU(3) structure vacua correspond to either complex (and with vanishing c_1) or symplectic manifolds, which are the two particular cases which inspired the definition of generalized Calabi–Yau. In the generic SU(3)×SU(3) case, vacua can be a complex–symplectic hybrid, namely a manifold that is locally a product of a complex k-fold times an $6 - 2k$ symplectic manifold.

Physically, the generalized Calabi–Yau condition has also been argued to imply the existence of a topological model [8, 9], not necessarily coming from the twisting of a $(2, 2)$ model, which generalizes the A and B models. In other words, we find that all $\mathcal{N} = 1$ Minkowski vacua have an underlying topological model. When there is a $(2, 2)$ model, both pure spinors are closed under $(d + H\wedge)$ [2], which reflects the fact that two topological models can be defined. This condition unifies the Calabi–Yau case and the $(2, 2)$ models of [10]. It corresponds to an unbroken $\mathcal{N} = 2$ in the target space, and it has recently been found from supergravity in [3]. Although the $\mathcal{N} = 2$ requirement of having two twisted closed pure spinors looks like our $\mathcal{N} = 1$ equations (1.1) for $F = A = 0$, we stress again that (1.1) applies only when the RR fluxes are non zero. Therefore we cannot obtain from there the $\mathcal{N} = 1$ equations for pure NS flux, which correspond to a $(2, 1)$ model.

Another feature of (1.1) is that they are essentially identical for IIA and IIB. This suggests there must exist some form of mirror symmetry for these compactifications exchanging the even and odd pure spinors [11–13]. As far as we know, mirror symmetry could even be present when supersymmetry is spontaneously broken [11, 13, 14]. For the case at hand of unbroken $\mathcal{N} = 1$, however, this is made particularly concrete by the remark above that all vacua have an underlying topological model; mirror symmetry has long been viewed [15] as an exchange of topological models, without necessarily involving Calabi–Yau's.

Presumably there are connections to more recent lines of thought relating Hitchin functionals [16] to topological theories [17–19]. Particularly promising seems the results in [20] about the quantization of the functional, which relate directly to generalized Calabi–Yau's with SU(3)×SU(3) structure.

On the practical side SU(3)×SU(3) structures allow to treat more easily cases which would be otherwise complicated, and that for this reason may have been regarded so far as pathological or unpractical. Most mathematical approaches, starting from [21], have to date focused on structure groups on T. For SU(3) structure the theory of intrinsic torsions involved is useful and manageable; already for SU(2), which is the intersection of two SU(3) structures, it becomes less tractable, as more representations appear in the game. Furthermore, there are cases in between SU(3) and SU(2) which do not really deserve a name as a structure on T, such as two SU(3) structures degenerating at some points. All these cases can be treated on the same footing if we consider structures on $T \oplus T^*$: they are SU(3)×SU(3) structures. The pure SU(3) structure case is rather the exception than the rule; with SU(3)×SU(3) structures more cases are available. An extreme example can be found in [22], where the structure is actually the trivial group: some nilmanifolds are complex, some are

symplectic, most are neither; but they all are generalized complex.

In the following sections we give a detailed explanation of the conditions to have $\mathcal{N}=1$ supersymmetry. In section 2 we review the algebraic conditions, describing the SU(3), SU(2) and generic SU(3)×SU(3) cases. We give the differential conditions in section 3, and discuss their implications. Section 4 contains a discussion of the connections to other results and applications. Then in appendix A we briefly review the relations between the pure spinor equations and topological models. We also propose, in appendix B, how to interpret the $\mathcal{N}=1$ condition for vacua without RR fluxes in terms of Courant brackets.

2 The algebraic part: structure groups

In this section we discuss how SU(3)×SU(3) structures on $T \oplus T^*$ describe in a unified way the various structures arising on T [1–3]. The real advantages of the SU(3)×SU(3) description will however emerge in the discussion of the differential conditions.

The supersymmetry transformations for type II theories contain two ten–dimensional Majorana-Weyl spinor parameters $\epsilon_{1,2}$. If the ten–dimensional manifold is topologically a homogeneous four–dimensional space (AdS, Minkowski) times an internal six–manifold, the ten-dimensional spinors can be decomposed into spinors in four dimensions times internal spinors. Since we are interested in backgrounds preserving four-dimensional $\mathcal{N}=1$ supersymmetry, there should be a single four dimensional conserved spinor. We therefore write

$$\begin{array}{rcl} \epsilon^1 &=& \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_- \,, \\ \epsilon^2 &=& \zeta_+ \otimes \eta^2_- + \zeta_- \otimes \eta^2_+ \,, \end{array} \quad (\mathrm{IIA}); \qquad \epsilon^i = \zeta_+ \otimes \eta^i_+ + \zeta_- \otimes \eta^i_- \,, \quad (\mathrm{IIB})\,, \qquad (2.1)$$

for any four–dimensional spinor ζ_+, with ζ_- being its Majorana conjugate[2]. Also, $(\eta^i_+)^* = \eta^i_-$, in such a way that ϵ^i are Majorana in ten dimensions. Here, we have not yet taken the spinors to be normalized to any particular value.

Given two spinors on the internal manifold, there are different possible relations among them, that lead to different structures.

To begin with, the spinors η^1 and η^2 may simply be proportional. Then they define what is called an *SU(3) structure* (this is uniquely defined by a nowhere vanishing spinor invariant under SU(3) ⊂ SO(6), not necessarily covariantly constant). A prominent example of an SU(3) structure manifold is a Calabi–Yau 3-fold. In that case, the invariant spinor is also covariantly constant and the structure group coincides with the holonomy group. The case of SU(3) structure is particularly simple – and hence much studied – due to the fact that few representations are involved. The manifold can be characterized either by the SU(3) invariant spinor, or by an SU(3) invariant real two-form J and a complex three-form Ω, obeying $J \wedge \Omega = 0$, $i\Omega \wedge \bar{\Omega} = \frac{4}{3} J^3$. However for several purposes it is better to deal with e^{iJ} rather with J alone. Already in the study of branes on Calabi–Yau manifolds, e^{iJ} emerges

[2] We could imagine a more general way of relating the two four–dimensional spinors in $\epsilon_{1,2}$, like for example $\xi_1 = A_{\mu\nu}\gamma^{\mu\nu}\xi_2$, but maximal symmetry in four dimensions is only compatible with (2.1); see for example [23].

as the mirror of Ω. This feature persists, at least at local level, for general SU(3) structure [11]. In [7], it was also shown that the $\mathcal{N}=1$ condition in supergravity naturally uses e^{iJ}. We will come back to this issue in the next subsection.

The other extreme choice for the relation between the spinors η^1 and η^2 is to take them never parallel. In this case there is a bilinear which defines a complex vector field without zeros

$$\eta^{1\,\dagger}_{+}\gamma_m\eta^2_{-} = v_m - i w_m \,. \tag{2.2}$$

The two spinors give rise to two different SU(3) structures whose intersection gives an *SU(2) structure*. Each SU(3) structure has an associated almost complex structure $J_m{}^p$.[3] The product of the two almost complex structures, $J_{1m}{}^p J_{2p}{}^n$, is a tensor which squares to 1 and has four negative and two positive eigenvalues; it is called an *almost product structure* and can be used to split the tangent space at every point (and hence also all the bundles $\Lambda^p T^*$) in four plus two dimensions[4]:

$$J_{1,2} = j \pm v\wedge w \ , \qquad \Omega_{1,2} = \omega \wedge (v \pm i w)\,. \tag{2.3}$$

The forms j, ω, v and w define the SU(2) structure in 6 dimensions. They are all nowhere vanishing. Alternatively, it is also possible to define the SU(2) structure from an SU(3) one, say J_1 and Ω_1, and one vector v.

Both SU(3) and SU(2) impose some topological constraints on the manifold. The one imposed by SU(2) is stronger: due to the fact that the two spinors are never parallel, the vector v in (2.3) is nowhere vanishing. This is of course possible if and only if the Euler characteristic χ of the manifold vanishes. Thus the condition $\chi(M)=0$ ensures the existence of a topological SU(2) structure on six-manifolds with SU(3) structure. On the contrary when the two spinors are parallel the vector defined in (2.2) vanishes everywhere.

There are however more general situations where the two spinors η^1 and η^2 can become parallel at points on the manifolds and this does not impose any extra topological constraint with respect to SU(3) structure. To treat all these cases in a uniform way it is better to consider structures on the sum of the tangent and cotangent bundles rather than on the tangent bundle alone.

2.1 Structures on $T \oplus T^*$

Enlarging the space by combining the tangent and cotangent bundles in a single bundle, $T\oplus T^*$, allows not only to give a unified description of the structures on T we mentioned above, but also to understand the mathematical meaning of the formal sum of forms e^{iJ} and what Ω and e^{iJ} have in common.

In general, a formal sum of forms can be viewed as a representation of O(6, 6), which is the structure group of $T\oplus T^*$. In fact, Ω and e^{iJ} transform under O(6, 6)

[3] For a given SU(3) structure, we call the two–form J and the almost complex structure $J_m{}^n$ induced by Ω with the same name. This should not lead to a confusion, for the complex structure the indexes will always be written explicitly.

[4] j and ω can be thought of as an SU(2) structure in four dimensions, and v and w as a trivial structure in two. This split however does not mean a priori that there is any four–dimensional submanifold along which j and ω are defined; this would mean the almost product structure is integrable.

in exactly the same fashion as the formal sum of RR fields transforms under the T–duality group. Moreover, they share another very important property – *purity* – which will be explained shortly.

Let us first consider the spinor group and the representation of the Clifford algebra corresponding to O$(6,6)$, which is called Clifford$(6,6)$. It is defined by

$$\{\lambda^m, \lambda^n\} = 0 \ , \qquad \{\lambda^m, \iota_n\} = \delta^m{}_n \ , \qquad \{\iota_m, \iota_n\} = 0 \ ,$$

where $\delta^m{}_n$ is the 6+6–dimensional metric $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ on $T \oplus T^*$. A useful representation is given by the six wedge operators and the six contractions

$$\lambda^m \equiv dx^m \wedge \qquad\qquad \iota_n \equiv \iota_{\partial_n} \ . \tag{2.4}$$

Starting from a Clifford vacuum one can generate any form by acting with the appropriate λ's and ι's. It is then clear why a formal sum of forms in $\Lambda^\bullet T^*$ is a Clifford$(6,6)$ spinor. There are irreducible ("Majorana-Weyl") representations of Spin(6,6), which correspond to real ("Majorana") even or odd forms ("Weyl"). Here we will work with Weyl Clifford(6,6) spinors $\Phi_\pm \in \Lambda^{even/odd}T^*$.

A Clifford(6,6) spinor is pure if there exist six linear combinations of the $\{\lambda^m, \iota_n\}$ which annihilate it.

We argued that a sum of forms is a Clifford$(6,6)$ spinor. It can nevertheless also be mapped to a bispinor, using the Clifford map:

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1\ldots i_k} dx^{i_i} \wedge \ldots \wedge dx^{i_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1\ldots i_k} \gamma^{i_i\ldots i_k}_{\alpha\beta} \ . \tag{2.5}$$

On a bispinor, we can act with usual gamma matrices from the left (which we denote as $\overrightarrow{\gamma}^m$) and from the right (which we denote as $\overleftarrow{\gamma}^m$)

$$\overrightarrow{\gamma}^m = \frac{1}{2}(\lambda^m + g^{mn}\iota_n) \ , \qquad \overleftarrow{\gamma}^m = \frac{1}{2}(\lambda^m \pm g^{mn}\iota_n) \ , \tag{2.6}$$

where the $\pm$ sign is the parity of the spinor on which $\overleftarrow{\gamma}^m$ acts. The result of this action is still a bispinor. This gives a map between two copies of Clifford(6) and Clifford$(6,6)$. This map was used extensively in [7] and independently in [3,4].

One can use the Clifford map (2.5) to determine the pure spinors on $T \oplus T^*$, Φ_+ and Φ_-, as the counter-image of bispinors, namely

$$\Phi_+ \equiv \eta^1_+ \otimes \eta^{2\,\dagger}_+ \ , \qquad \Phi_- \equiv \eta^1_+ \otimes \eta^{2\,\dagger}_- \ . \tag{2.7}$$

The bispinor picture is useful to check that these spinors are pure. The six linear combinations of the $\{\lambda^m, \iota_n\}$ required to show purity of both $\Phi_\pm$ are the counter-image under (2.6) of ordinary gamma matrices, three acting from the left and three from the right[5] :

$$(\delta + iJ_1)_m{}^n \gamma_n \, \eta^1_+ \otimes \eta^{2\,\dagger}_\pm = 0 \ , \qquad \eta^1_+ \otimes \eta^{2\,\dagger}_\pm \gamma_n (\delta \mp iJ_2)_m{}^n = 0 \ . \tag{2.8}$$

[5]The fact that each of the spinors η^i_+ has three annihilators among the six gamma matrices can be rephrased in terms of purity for Clifford(6) spinors: all spinors in six dimensions are pure (see for example [27]).

The purity condition can also be reformulated in terms of structure groups on $T \oplus T^*$: each pure spinor reduces the structure group from $\mathrm{O}(6,6)$ to $\mathrm{SU}(3,3)$. Furthermore, if two given pure spinors satisfy a compatibility condition, namely they have exactly three common annihilators, the structure is reduced to SU(3)×SU(3). From eq. (2.8) it is easy to see that $\eta^1_+ \otimes \eta^{2\dagger}_\pm$ are compatible: the three annihilators on the left are common to both pure spinors. $\Phi_\pm$ define therefore an SU(3)×SU(3) structure on $T \oplus T^*$.

We will now look again at the particular cases of SU(3)×SU(3) structures introduced before, and give the corresponding Clifford(6,6) pure spinors.

2.1.1 Pure spinors for SU(3) structure on T

If the manifold has SU(3) structure there are two pure spinors that one can build with the forms J and Ω. These are precisely e^{iJ} and Ω. In terms of the invariant spinors on the manifold we can define these as special cases of (2.7)

$$\hat{\Phi}_+ = \frac{1}{8}e^{-iJ}\,, \qquad \hat{\Phi}_- = -\frac{i}{8}\Omega\,. \tag{2.9}$$

Differently from (2.7), the pure spinors $\hat{\Phi}_\pm$ are normalized, that is to say that they are built out of a single spinor $\eta = \eta_1 = \eta_2$ normalized to one; in the future we shall reserve the hats for the normalized quantities.

From (2.8) we can check that both Ω and e^{iJ} are indeed pure

$$\begin{array}{c} \lambda^i\Omega = 0\ , \\ \iota_{\bar{i}}\Omega = 0 \end{array} \quad ; \qquad\qquad (\iota_m - iJ_{mn}\lambda^n)e^{-iJ} = 0\ , \tag{2.10}$$

and that they are compatible: the three common annihilators are $\iota_{\bar{i}} - iJ_{\bar{i}j}\lambda^j$. Hence the pair (e^{iJ}, Ω) reduces the structure on $T \oplus T^*$ to SU(3)×SU(3).

2.1.2 Pure spinors for SU(2) structure on T

SU(2) structure on T is also a particular case of SU(3)×SU(3). η^1 and η^2 are never parallel. For example, when they are everywhere orthogonal, they define what is known as a static SU(2) structure. In this case, the two compatible pure spinors of (2.8) are

$$\hat{\Phi}_+ = e^{-i\,j} \wedge (v + iw)\ , \qquad \hat{\Phi}_- = \omega \wedge e^{-i\,v\wedge w}\ , \tag{2.11}$$

in terms of the almost product structure introduced above (2.3). Notice that both have a kind of four–two split: the first looks like e^{iJ} in four dimensions and like Ω in two dimensions; the second like Ω in four dimensions and like e^{iJ} in two dimensions.

2.1.3 The generic SU(3)×SU(3) case on $T \oplus T^*$

We have detailed so far two cases of SU(3)×SU(3) structures on $T \oplus T^*$: SU(3) on T and SU(2) on T. In the first case, the two spinors are proportional; in the second case, they are never parallel. As already mentioned, in the generic case they can be parallel at some points.

For this generic case, let us first write the spinor η^2_+ as a linear combination of elements in a basis derived from η^1. This basis for Clifford(6) spinors is the usual one obtained acting on a Clifford vacuum η^1_+: $\eta^1_+, \gamma^m\eta^1_+, \gamma^m\eta^1_-, \eta^1_-$ (note that by purity of the spinor η^1, only three out of the six gamma matrices give a nonzero spinor). To have positive chirality η^2_+ can only be a linear combination of η^1_+ and $\gamma^m\eta^1_-$; one can write, in a similar notation as in [3],

$$\eta^2_+ = c\,\eta^1_+ + (v+iw)\cdot\eta^1_- \tag{2.12}$$

where $(v+iw)\cdot$ is the Clifford action $(v+iw)_m\gamma^m$, and c is a complex function. Using this, equations (2.9) and the four–two split given in (2.3), one finds[6] [3]

$$\hat{\Phi}_+ = \frac{1}{8}(\bar{c}\,e^{-i\,j} - i\omega)\wedge e^{-iv\wedge w}\ , \qquad \hat{\Phi}_- = -\frac{1}{8}(e^{-i\,j} + ic\,\omega)\wedge(v+iw)\ . \tag{2.13}$$

These formulae are written using the *local* SU(2) structure defined by the pair of spinors or equivalently the intersection of two SU(3) structures. The spinors in (2.13) are the intrinsic invariant objects of the SU(3)×SU(3) structure, while the pure spinors defined by $\eta^i_+\otimes\eta^{i\,\dagger}_\pm$ are invariant under only one SU(3). No particular properties of j, ω, v and w are assumed - the structure group of $T\oplus T^*$ being SU(3)×SU(3) simply tells us that $\hat{\Phi}_+$ and $\hat{\Phi}_-$ are well defined and nowhere vanishing[7]. If v and w turn out to be nowhere-vanishing, the internal space M is more restricted than we have generally assumed (namely besides admitting an almost complex structure and having a vanishing first Chern class, it should have $\chi = 0$).

Let us summarize the normalized pure spinors in the various cases:

	$\hat{\Phi}_+$	$\hat{\Phi}_-$
SU(3)	e^{-iJ}	Ω
SU(2)	$\omega\wedge e^{-i\,v\wedge w}$	$e^{-i\,j}\wedge(v+iw)$
SU(3)×SU(3)	$(\bar{c}\,e^{-i\,j} - i\omega)\wedge e^{-iv\wedge w}$	$-(e^{-i\,j} + ic\,\omega)\wedge(v+iw)$

Table 1: normalized pure spinors for the various structures

We remind the reader that here we have presented the normalized pure spinors (c and the norm of the complex vector should be related by $|c|^2 + |v+iw|^2 = 1$). The differential equations for $\mathcal{N}=1$ supersymmetry given in the next section will be for their non-normalized counterparts.

The analysis of this section only regards topological properties of the manifold. Much more meaningful constraints are obtained when considering the integrability properties of the structures considered so far. This is the topic of the next section.

[6] We are using that the SU(3) structure defined by η^1 is given by $J = j + v\wedge w$, $\Omega = \omega\wedge(v+iw)$.

[7] At the points where the two spinors η^1, η^2 become parallel, i.e. when $(v+iw) = 0$, the lower case forms j and ω are not well defined, but we should think of the products $e^{i(j+v\wedge w)}$ and $\omega\wedge(v+iw)$ as e^{iJ} and Ω (the two pure bispinors built out of η^1), which are well defined.

3 The differential part: integrability

SU(3)×SU(3) structures describe in a unified way several types of structures on T. In this section we will see that the conditions for $\mathcal{N}=1$ vacua translate into conditions for the integrability of an SU(3)×SU(3) structure.

We will first present the equations, then we give a brief explanation of how they were obtained and finally discuss their mathematical interpretation.

Preserved supersymmetry imposes differential equations on the Clifford(6) spinors. As a consequence, the pure Clifford(6,6) spinors, given in (2.7) as bispinors and in (2.13) as formal sums of forms, have to obey certain differential conditions. In order to preserve $\mathcal{N}=1$ supersymmetry, the conditions are

$$\begin{aligned} e^{-2A+\phi}(d+H\wedge)(e^{2A-\phi}\Phi_+) &= 2\mu\, e^{-A}\,\mathrm{Re}(\Phi_-)\,, && (3.1)\\ e^{-2A+\phi}(d+H\wedge)(e^{2A-\phi}\Phi_-) &= 3i\, e^{-A}\,\mathrm{Im}(\bar\mu\Phi_+)+dA\wedge\bar\Phi_- && (3.2)\\ &\quad -\frac{1}{16}e^{\phi}\Big[(|a|^2-|b|^2)F_{\mathrm{IIA}\,-}+i(|a|^2+|b|^2)*F_{\mathrm{IIA}\,+}\Big]\,, \end{aligned}$$

for type IIA, and

$$\begin{aligned} e^{-2A+\phi}(d-H\wedge)(e^{2A-\phi}\Phi_+) &= -3i\, e^{-A}\,\mathrm{Im}(\bar\mu\Phi_-)+dA\wedge\bar\Phi_+ && (3.3)\\ &\quad +\frac{1}{16}e^{\phi}\Big[(|a|^2-|b|^2)F_{\mathrm{IIB}\,+}+i(|a|^2+|b|^2)*F_{\mathrm{IIB}\,-}\Big]\,,\\ e^{-2A+\phi}(d-H\wedge)(e^{2A-\phi}\Phi_-) &= -2\mu\, e^{-A}\,\mathrm{Re}(\Phi_+)\,, && (3.4) \end{aligned}$$

for type IIB. Let us explain the various pieces in these equations: in both theories F is a formal sum of modified RR fluxes (i. e. obeying a non standard Bianchi identity $dF_n = H\wedge F_{n-3}$) [8,9]

$$F_{\mathrm{IIA}\,\pm} = F_0 \pm F_2 + F_4 \pm F_6\ , \qquad F_{\mathrm{IIB}\,\pm} = F_1 \pm F_3 + F_5\ . \tag{3.5}$$

A is the warp factor in a warped product metric of the form

$$ds^2 = e^{2A} g_{\mu\nu}dx^\mu dx^\nu + ds^2_6 \tag{3.6}$$

where $g_{\mu\nu}$ is a maximally symmetric space with zero or negative cosmological constant Λ (Minkowski, or AdS). Λ is related to the complex quantity μ by

$$\Lambda = -|\mu|^2\,. \tag{3.7}$$

Finally,

$$|a|^2 = |\eta_1^+|^2\,, \qquad |b|^2 = |\eta_2^+|^2 = |a|^2(|c|^2+|v+iw|^2)\,, \tag{3.8}$$

where in the last equality we have used (2.12). From (2.7) we can see that the norms of the pure spinors are also given in terms of a and b by

$$|\Phi_\pm|^2 = |a|^2|b|^2\,. \tag{3.9}$$

[8]All fluxes here are internal, any piece along spacetime has been traded for a dual internal flux. See the end of this section for more details.

[9]We thank L. Martucci and P. Smyth for correcting signs in an earlier version of (3.2,3.3).

$\mathcal{N}=1$ supersymmetry imposes the following relation between the norms

$$d|a|^2 = |b|^2 dA \ , \qquad d|b|^2 = |a|^2 dA \ , \tag{3.10}$$

for both IIA and IIB.

A very important remark is in order: equations (3.1-3.4) do not cover the case $F=0$, $\mathcal{N}=1$ vacua. From the Hodge decomposition (3.16), it is easy to see that setting $F=0$ implies $dA=0$. Then (3.10) tells that a and b have constant norms. If both of them are non-zero, this results into two independent supersymmetry parameters in four dimensions. Therefore equations (3.1-3.4) describe an $\mathcal{N}=2$ rather than an $\mathcal{N}=1$ vacuum. In order to have $\mathcal{N}=1$ vacua with only NS flux one of the functions a and b should be zero. In this case both pure spinors $\Phi_\pm$ are zero, and our equations still hold but do not contain much information. However the supersymmetry conditions for $\mathcal{N}=1$ vacua with only NS flux are well known [24, 25, 28]. The structure has to be SU(3), since there one spinor is involved, and it has to obey $(d+H\wedge)(e^{2\phi}\Omega)=0$, $e^{2\phi}d(e^{-2\phi}J)=*H$ and $d(e^{2\phi}J^2)=0$. It is not possible to write an equation for e^{iJ} of the same form as (3.1). Nevertheless it is possible to pack all the choices for a and b in a single equation. However, in this case the action of H is more complicated than in (3.1-3.4) (see [7]) and at the moment lacks of a mathematical meaning, unlike the twist $d+H\wedge$, whose significance will be reviewed in section 3.3.

As we will discuss in detail in section 3.2, equations (3.1),(3.2) (or (3.3),(3.4)) together with (3.10) are *necessary and sufficient* to find a solution to the $\mathcal{N}=1$ supersymmetry conditions. This means that these equations contain exactly the same amount of information than the original supersymmetry variations $\delta_\epsilon \psi_M = 0$ and $\delta_\epsilon \lambda = 0$ with ϵ given in (2.1). To find a vacuum, one has to supplement preserved supersymmetry conditions with Bianchi identities and the equations of motion for the fluxes [6]. In this paper we do not address this issue.

Let us now explain briefly how (3.1-3.4) were obtained. For more details, see [7], as the method is very similar. To set the conventions, we use the string frame and the democratic formulation of [29]. This formulation considers all RR fluxes $F_{0,2,4,6,8}$ for IIA and $F_{1,3,5,7,9}$ for IIB, obeying a duality condition $F_n = (-1)^{Int[n/2]} * F_{10-n}$. Since we are interested in a maximally symmetric four-dimensional space-time, we only turn on fluxes that have either none or four components along it. For a flux with four legs along space-time, we can use the duality relation to write it in terms of an internal flux. For example, $F_{\lambda\mu\nu\rho}$ can be traded for F_6 fully along internal space. As a consequence, all fluxes in (3.1-3.4) are internal. Secondly, for the supersymmetry parameters we use the decomposition given in (2.1).

We start by using the bispinor form of the pure spinors $\Phi_\pm$ given in (2.7). This allows to write the exterior derivative of the Clifford$(6,6)$ spinors in terms of the covariant derivative of the Clifford(6) spinors, namely

$$d\Phi_\pm = dx^m \wedge \nabla_m \Phi_\pm = dx^m \wedge \Big((\nabla_m \eta^1_+) \otimes \eta^{2\dagger}_\pm + \eta^1_+ \otimes (\nabla_m \eta^{2\dagger}_\pm) \Big) \ . \tag{3.11}$$

The internal component of the supersymmetry variation of the gravitino, $\delta\psi_m = 0$, gives the covariant derivative of the spinors in terms of the fluxes. The equations are considerably simplified if we use additionally the dilatino variation $\delta\lambda$ and space-time gravitino variation $\delta\psi_\mu = 0$. The use of dilatino variation is the reason for the appearance of derivatives of the dilaton in (3.1-3.4), while using the space-time gravitino variation introduces the cosmological constant μ. The latter appears in the covariant derivative of the supersymmetry parameter along space-time. To be more precise, there are two possible Killing spinor equations for constant negative curvature spaces, namely $\nabla_\mu\zeta = \frac{\mu_1}{2}\gamma_\mu\zeta$ and $\nabla_\mu\zeta = i\frac{\mu_2}{2}\gamma_\mu\gamma_5\zeta$. We have kept both of them, as done for example in [6, 23, 30], and defined the complex quantity $\mu = \mu_1 + i\mu_2$. This complex quantity has the interpretation in four dimensions as a vacuum superpotential [31]. The cosmological constant is determined by the norm of μ as given in (3.7).

3.1 Intrinsic torsions for SU(3) and SU(3)×SU(3) structures

In order to prove that the pure spinor equations contain the same information as the supersymmetry variations, we have to set up the basic machinery for the SU(3)×SU(3) torsions. The way we do this is simply to compare it with the better known case of SU(3).

As just said, we start with the case of SU(3) structure, one of the defining features of which is the existence of a nowhere vanishing invariant spinor. In general this spinor is not covariantly constant, but using a bit of SU(3) group theory we can write the covariant derivative as

$$\nabla_m\eta = i\,q_m\gamma_7\,\eta + i\,q_{mn}\gamma^n\eta\,, \tag{3.12}$$

where $\gamma_7 = -\frac{i}{6!}\epsilon_{mnpqrs}\gamma^{mnpqrs}$. Equivalently, the invariant forms J and Ω are not closed

$$\begin{aligned} dJ &= -\tfrac{3}{2}\,\mathrm{Im}(W_1\bar{\Omega}) + W_4\wedge J + W_3\,, \\ d\Omega &= W_1 J^2 + W_2\wedge J + \overline{W}_5\wedge\Omega\ \ . \end{aligned} \tag{3.13}$$

The covariant derivative of the spinor contains the same information as the exterior derivatives dJ and $d\Omega$

$$(q_m,\, q_{mn}) \leftrightarrow W_i\,,$$

and the precise formulae for this map can be found in [7, 11].

We are ready to give the analogue of these formulae for the SU(3)×SU(3) structure. Let us start by considering the covariant derivative of the spinors. By expanding $\nabla_m\eta^i_\pm$ in a basis of spinors, we can define the intrinsic torsions:

$$\nabla_m\eta^i_+ = iq^i_m\eta^i_+ + iq^i_{mn}\gamma^n\eta^i_-\,, \tag{3.14}$$

which simply duplicates the usual SU(3) structure definition (3.12). We can use this to compute the exterior derivatives of $\Phi_\pm$, thus generalizing the W_i of SU(3)

structures:

$$
\begin{aligned}
d\Phi_+ &= W^{10}_m \gamma^m \Phi_+ + W^{01}_m \Phi_+ \gamma^m + W^{30} \bar\Phi_- + W^{21}_{mn} \gamma^m \bar\Phi_- \gamma^n + W^{12}_{mn} \gamma^m \Phi_- \gamma^n + W^{03} \Phi_- \,, \\
d\Phi_- &= W^{13}_m \gamma^m \Phi_- + W^{02}_m \Phi_- \gamma^m + W^{33} \bar\Phi_+ + W^{22}_{mn} \gamma^m \bar\Phi_+ \gamma^n + W^{11}_{mn} \gamma^m \Phi_+ \gamma^n + W^{00} \Phi_+ \,.
\end{aligned}
\tag{3.15}
$$

The change of basis between (3.15) and (3.14) can easily be found and generalizes the one between (3.12) and (3.13); the labeling of the components of the intrinsic torsion will become clear in a moment.

Since the formal sums of forms are a representation of O(6, 6) of which SU(3)×SU(3) is a subgroup, we can decompose the forms under SU(3)×SU(3). The decomposition is given by the following basis [2, 32], analogous to the Hodge diamond:

$$
\begin{array}{ccccccc}
 & & & \Phi_+ & & & \\
 & & \gamma^m\Phi_+ & & \Phi_+\gamma^m & & \\
 & \gamma^m\bar\Phi_- & & \gamma^m\Phi_+\gamma^n & & \Phi_-\gamma^m & \\
\bar\Phi_- & & \gamma^m\bar\Phi_-\gamma^n & & \gamma^m\Phi_-\gamma^n & & \Phi_- \\
 & \bar\Phi_-\gamma^m & & \gamma^m\bar\Phi_+\gamma^n & & \gamma^m\Phi_- & \\
 & & \bar\Phi_+\gamma^m & & \gamma^m\bar\Phi_+ & & \\
 & & & \bar\Phi_+ & & &
\end{array}
\tag{3.16}
$$

Remember that only three of the six γ^m survive in each of these expressions; for example, since $\gamma^m\eta_1^+ = \Pi_1^{mn}\gamma_n\eta_1^+$, one has $\gamma^m\Phi_\pm = \Pi_1^{mn}\gamma_n\Phi_\pm$, where Π_1 is the holomorphic projector with respect to the almost complex structure J_1. Each entry of this Hodge diamond should be understood as a representation of SU(3)×SU(3): from the top, $(1,1)$, $(3,1)$, $(1,3)$ and so on.

Returning to (3.15), we can see now that the superscripts on W^{ij} refer to the position of the summand in the Hodge diamond (3.16), with the top element being marked as 00, the second row – 10, 01 and so on. We can also notice a few things about the new SU(3)×SU(3) intrinsic torsions. First of all, each of them only contains terms at distance at most three, horizontally or vertically, in the Hodge decomposition (3.16). This generalizes the fact that $d\Omega$, for SU(3) structures, contains a form of degree $(3,1)$ and (if the almost complex structure is not integrable) forms of degree $(2,2)$, but no form of degree $(1,3)$. There are additional relations between the W's: $W^{30} = W^{33}$, $W^{03} = \overline{W^{00}}$. These are a generalization of the fact that for SU(3) structures, W_1 is contained both in dJ and $d\Omega$. Both these relations follow from the change of basis between (3.14) and (3.15).[10]

3.2 Pure spinor equations vs. supersymmetry

We can now finally prove the equivalence of (3.1 - 3.4) and (3.10) to the vanishing of the supersymmetry variations, for which we will use a shorthand "sufficiency". Our strategy is simply to count the number of independent equations for each

[10] They can be obtained by deriving the compatibility condition, which can be rephrased as $\mathrm{Tr}(\Phi^\dagger(\lambda^m)\Phi) = \mathrm{Tr}(\Phi^\dagger(\iota^m\Phi)) = 0$ (with Φ any of the Clifford vacua in 3.16) and using the constraints of normalization, $\mathrm{Tr}(\Phi_+^\dagger\Phi_+) = \mathrm{Tr}(\Phi_-^\dagger\Phi_-)$.

SU(3)×SU(3) representation that we get from the original supersymmetry variations $\delta\psi_M = 0 = \delta\lambda$ (which take values in Clifford(6)), and to compare with those obtained from the set (3.1 – 3.4) and (3.10) (which are Clifford(6,6)-valued).

To perform the SU(3)×SU(3) decomposition on both sides (Clifford(6) and Clifford(6,6)) we use the fact that any even or odd real form $F_\pm$ can be expanded in the SU(3)×SU(3) basis given in (3.16) in the following way

$$F_\pm = F^0\Phi_\pm + F^1_m\gamma^m\bar{\Phi}_\mp + F_{mn}\gamma^m\Phi_\pm\gamma^n + F^2_m\Phi_\mp\gamma^m \ + (\text{c. c.})\,. \tag{3.17}$$

On the Clifford(6) side, we insert the expansion (3.17) for the NS and RR fluxes in the supersymmetry variations $\delta_\epsilon\psi_M = 0 = \delta_\epsilon\lambda$. These equations contain in addition the covariant derivative of the internal spinor, for which we use (3.14). This relates q^i_m to F^i_m and H^i_m (the components of RR and NS flux in their expansion à la (3.17)) and q^i_{mn} to F_{mn},F^0,H_{mn} and H^0. We are not going to give the explicit expressions here, as they are not particularly enlightening. What we are interested in is the number of independent equations that we get for each SU(3)×SU(3) representation. There are four independent equations for quantities without indices (in the (1,1)), eight for quantities with one index only (in the $(3,1)$, $(1,3)$, $(\bar{3},1)$ and $(1,\bar{3})$) and four for quantities with two indices ($(3,3)$, $(\bar{3},3)$ and so on).

On the Clifford(6,6) side, we plug the decomposition of the RR and NS fluxes (3.17) in (3.1-3.4). For the derivative of the pure spinor, we use (3.15) (remember that there is a one to one correspondence between $\{W^{ij}\}$ and (q^i_m, q^i_{mn}), so we could use the q's for torsions as well). As a result, we get again a set of equations for each SU(3)×SU(3) representation. After supplementing these with the equations for the norm of the pure spinors (3.10), we confirm that the number of independent equations is the same as on the Clifford(6) side (and of course they agree with the latter).

We therefore conclude that (3.1 - 3.4) and (3.10) are the necessary and sufficient conditions dictated by $\mathcal{N} = 1$ supersymmetry (again, these a priori are not enough to give solutions to the equations of motion since Bianchi identities and the equations of motion for the fluxes have still to be imposed).

Note that this procedure gives all the intrinsic torsion components W^{ij} in terms of fluxes, derivatives of the dilaton and warp factor. We do not write the explicit expressions here, but simply point out that they are very similar to the ones given in [7] for the pure SU(3) structure case.

3.3 Integrability for pure spinors and $\mathcal{N} = 1$ vacua

Equations (3.1) and (3.4) can be interpreted as an integrability condition for the SU(3)×SU(3) structure in IIA and IIB, respectively.

The standard way of introducing an integrability condition involves defining *generalized (almost) complex structures* $\mathcal{J}$. This is like an almost complex structure, except that it lives on $T \oplus T^*$ rather than on T. The existence of $\mathcal{J}$ reduces the structure group of $T \oplus T^*$ to $\mathrm{U}(3,3)$. Two compatible generalised almost complex

structures[11] reduce the structure group to U(3)×U(3). This closely parallels the discussion for pure spinors in the previous section; and in fact there is a one to one correspondance between an almost generalized complex structure and a pure spinor [1, 2]. Very briefly, consider a single pure spinor Φ. By definition, its annihilator is a subbundle of $T \oplus T^*$ of dimension six. It is always possible to find a generalised almost complex structure that has this annihilator as its $+$i eigenbundle.

The integrability condition then states that the annihilators of the pure spinor be invariant under a generalisation of the Lie bracket to $T \oplus T^*$, called the Courant bracket. Suppose that A is an element of $T \oplus T^*$, a linear combination of the $\{\lambda, \iota\}$ which generate the Clifford(6, 6) algebra. It can be considered as an operator on formal sums of forms. Given two such objects, we can produce a third one by

$$[A, B]_C(\omega) \equiv d(AB\omega) + Ad(B\omega) - Bd(A\omega) - BAd(\omega) - (A \leftrightarrow B) \tag{3.18}$$

where ω is a differential form. The operation $[\,,\,]_C$ is the Courant bracket.[12]

Let us see what the integrability condition for the annihilator implies on the pure spinor Φ. If we take ω to be Φ and A and B both in its annihilator, all terms but the last one go away. If we take $d\Phi = 0$, even the last term goes away and $[A, B]_C\Phi = 0$ – that is, $[A, B]_C$ is in the annihilator of Φ.

So we have shown that $d\Phi = 0$ is sufficient for the annihilator of Φ to be closed under the Courant bracket. (The necessary condition is only slightly more complicated; $d\Phi$ has to belong to the first Clifford level, that is, it has to be of the form $C\Phi$ for some C in $T \oplus T^*$.)

Actually, in the definition of the Courant bracket, d could be replaced just by any derivation operator. In particular, one can consider in (3.18) the operation $d + H\wedge$, where H is a closed 3-form, leading to the twisted Courant bracket. If $(d + H\wedge)\Phi = 0$, then the annihilator of Φ is closed under the twisted Courant bracket. Therfore, the first equation in (1.1), $(d + H\wedge)\Phi_1 = 0$, tells us that the annihilator of Φ_1 is closed under the twisted Courant bracket, or equivalently that the generalized almost complex structure associated to Φ_1 is integrable. Notice that $d + H\wedge$ this is the only combination of d and H_{mnp} which is a differential. It has already been considered in mathematics under the name of twisting [1].

As we will discuss in Appendix A, this equation has also appeared in the context of topological strings. It is somewhat more curious that also the equation involving RR fields has a similar structure and involves $(d + H\wedge)\Phi_2$ thus making the RR contribution play the role of a defect of integrability for the second pure spinor.

3.4 Complex-symplectic hybrids

We are finally ready to see what kind of manifolds are suitable for type II compactifications with resulting $\mathcal{N} = 1$ vacua. We will set the cosmological constant $\mu = 0$ in this section and for most of the next one, commenting briefly on the case $\mu \neq 0$

[11] Two generalised almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible if $[\mathcal{J}_1, \mathcal{J}_2] = 0$ and $\mathcal{J}_1\mathcal{J}_2 = G$, with G a positive definite metric on $T \oplus T^*$.

[12] This definition is more generally known as *derived bracket* [33]. Other particular cases include the Lie bracket on vectors, and the Schouten–Nijenhuis on multivectors.

at the end of that section. As clear from the general form of equations (3.1 - 3.4) a twisted closed pure spinor is required and thus the internal space must be a generalized Calabi-Yau. This still leaves plenty of possibilities as far as the differential–geometric structure of the internal space is concerned. In order to analyze this, it is useful to return to Table 1 and compare the different cases from a new perspective. For this we will need to introduce the notion of the type of a pure spinor.

In a regular neighborhood of the six-dimensional manifold, a pure spinor can be decomposed in the following way [2]

$$\Phi = e^A \wedge \omega_k \,, \tag{3.19}$$

where A is a complex 2-form and ω_k is a holomorphic k-form ($0 \leq k \leq 3$), which together obey $A^{6-2k} \wedge \omega_k \wedge \bar{\omega}_k \neq 0$. The number k is called the *type* of the pure spinor. It can also be obtained by counting the number of annihilators which can be expressed as pure λ^m – the intersection of the annihilator with T.

The usefulness of this definition is that it gives a local characterization of a generalized Calabi–Yau. Thanks of to a generalized Darboux theorem [2], locally we can always introduce a set of holomorphic coordinates $z^1, \ldots, z^k$, augmented by real ones $x^{2k+1}, \ldots, x^6$: the neighborhood of every point is isomorphic to $\mathbb{C}^k \times \mathbb{R}^{6-2k}$.

Given this definition, we can see from the Table 1 that the type of e^{iJ} is 0, and the type of Ω is 3 (this can also be obtained using the definition of the type in terms of the annihilators). Likewise, the types of the SU(2) pure spinors (2.11) are 2 and 1 for Φ_+ and Φ_- respectively. Generically, the type of a pure spinor is as low as allowed by parity. This is because imposing that the intersection of the annihilator with the tangent have a certain dimension is like imposing an equation. For example, an odd pure spinor will have generically type 1, and may have type 3 in some loci. So to have a pure spinor of type 3 everywhere, such as Ω, is very much non–generic.

The type of generic spinors for SU(3)×SU(3) can be seen by using (3.19) and slightly rewriting (2.13). We use $j \wedge \omega = \omega^2 = 0$ in order to obtain

$$\hat{\Phi}_+ = \frac{1}{8}\bar{c}e^{-i(j+v\wedge w+\frac{1}{c}\omega)} \,, \qquad \hat{\Phi}_- = -\frac{1}{8}(v+iw)\wedge e^{i(j+c\omega)} \,.$$

We see that when $c \neq 0$, the first pure spinor is a type 0 (symplectic) spinor, which jumps to type 2 at the points where c vanishes. On the other hand, when $v + iw \neq 0$, the second pure spinor is of type 1, and it jumps to type 3 at the points where the complex vector vanishes.

Returning to the pure spinor equations (3.1 – 3.4), again in the case $\mu = 0$, we see that the integrable spinor for IIA is given by Φ_+ and for IIB by Φ_-. (Note that parity of the integrable pure spinor coincides with the parity of the RR fluxes in each theory -even in IIA, odd in IIB-). Thus IIA generally prefers symplectic manifolds, and can admit hybrid types when $c = 0$, which corresponds to the internal spinors being orthogonal; this situation is usually referred to as a static structure. As for IIB - the situation is reversed, and generally the manifold M is of hybrid type (with one complex dimensional part), while at the special points of "pure" SU(3) structure, i. e. two spinors are proportional and the vector vanishes, M has to be complex [13].

[13] It was already noticed in [35] that a IIB solution on a manifold with SU(2) structure is not necessarily complex.

4 Discussion

The necessary conditions for $\mathcal{N}=1$ supersymmetry on six manifolds M with SU(3) $\times$ SU(3) structure on $T \oplus T^*$ boil down to a pair of equations for two pure spinors: one tells us that M is a twisted generalized Calabi–Yau, while the second one says that the combined RR fields act a source for Nijenhuis tensor. The twisted generalized Calabi–Yau has in IIA an integrable structure that is symplectic around regular points, and jumps to an hybrid complex–symplectic (with four and two real dimensions) at points where the structure is a static SU(2), namely where the two spinors are orthogonal. In IIB, on the contrary, the generalized Calabi–Yau has at regular points an integrable hybrid complex–symplectic (with two and four real dimensions). This integrable structure is purely complex at points where the structure becomes pure SU(3), i.e. when the two spinors are parallel.

The equations are similar to those found for topological models [8, 9] and $\mathcal{N}=2$ supersymmetry without RR fluxes [3]. In the latter case there is a worldsheet description with $(2,2)$ supersymmetry [10], and the two pure spinors are integrable. The corresponding schematic equations can be found in the table below:

	$\mathcal{N}=1$ (RR$\neq 0$)	$\mathcal{N}=2$ (RR$=0$)	top. model
SU(3)$\times$SU(3)	$(d+H\wedge)\Phi_1=0$ $(d+H\wedge)\Phi_2=F$	$(d+H\wedge)\Phi_1=0$ $(d+H\wedge)\Phi_2=0$	$(d+H\wedge)\Phi=0$

The first entry in this table is the pair of equations obtained in this paper. The other two deserve some comments.

First of all, as we mentioned after (3.10), the case in which $F=0$ and $\mathcal{N}=1$ is not covered by the analysis in this paper; that limit happens to give $\mathcal{N}=2$ vacua. Indeed, the equations in the second column describe what is called in [2] a "generalized Calabi–Yau metric" (stronger than a generalized Calabi-Yau manifold [1] used in this paper). In [2], these conditions were found to describe all $(2,2)$ nonlinear sigma models: both the usual Calabi–Yau case and also the models with twisted multiplets found in [10]. $(2,2)$ models correspond to $\mathcal{N}=2$ in the target space, and indeed the same equations were found to be implied by $\mathcal{N}=2$ vacua in supergravity. It is now also clear why $\mathcal{N}=1$ with NS only could not be described by the $(d+H\wedge)$ closure of two pure spinors: [2] finds that this would imply the existence of a $(2,2)$ sigma model, whereas $\mathcal{N}=1$ should correspond to a $(2,1)$ model only. We expand on this in Appendix B.

The last entry in the table is more interesting to us. The condition that *one* pure spinor be closed is sufficient for a topological model to exist on the manifold. This has been argued in [8, 9]; we will say more about it in Appendix A. For now, we can point out that the usual topological models for SU(3) structure, the A and B models, are obviously particular cases: the A model requires the two–form being closed, which implies $de^{iJ}=0$; the B model requires the manifold be complex, which is implied by $d\Omega=0$.

We would like to point out that the condition for having a topological model is, in the Minkowski case, one of the two equations we have found, (3.1) or (3.4). Hence in all the Minkowski vacua a topological model can be found.

At this point, however, we should add a word of caution. As mentioned many times, we have spelled out the $\mathcal{N}=1$ supersymmetry conditions only, but Bianchi identity is still to be imposed. One of the consequences of it are the well–known no–go theorem constraining the possibility of finding *compact* examples. In the present context, it is easy to see where the problem comes from. For Minkowski ($\mu = 0$), our equations say essentially $F = d\Phi$. Bianchi identities together with the flux equations of motion say that F is a harmonic form. These two statements together imply that $F = 0$ on a compact manifold. Of course there are many noncompact solutions, and they all will fall in our classification. However finding interesting compact examples is clearly of some importance. There are some ways of avoiding the no-go theorems which typically involve leaving the supergravity approximation and including sources or quantum corrections.

These problems are not expected to arise in AdS compactifications. While the equations (3.1 – 3.4) included a cosmological constant μ in the analysis of the equations this was set to zero. It is not hard to see that when $\mu \neq 0$ the generalized Calabi–Yau condition is violated. Moreover we get a mixing between the two pure spinors. In situations where μ is induced by a flux through the four-dimensional spacetime, with all the other fluxes turned off, we get an extension of the nearly-Kähler geometry corresponding to pure SU(3) structure, which has $d\Omega = \mu J^2$. While the case of $\mu \neq 0$ did not receive much attention here, it presents readily available examples of compact manifolds, where in particular Bianchi identities are solved without introducing orientifolds planes or other complications.

We will finish with two speculations. One practical advantage of having found the underlying geometry of $\mathcal{N}=1$ vacua might be a systematic approach to the problem of counting moduli. Deformations of generalized Calabi–Yau structures have already been studied in [2]. These are yet to be completed by the second condition involving the RR fluxes, and in principle a coupled system needs to be analyzed. We may just observe at this point that the form of the second equation ((3.2) for IIA, (3.3) for IIB) suggests that a deformation of Φ might be simply offset by a deformation of the potential C. We hope to return on the subject soon.

Finally, these equations might suggest ways to find a worldsheet realization of RR backgrounds. This is admittedly a long shot, but consider the following. First of all, there has been progress towards obtaining models with only *one* $(d + H\wedge)$ closed pure spinor. Second, the fact that we always have a topological model associated with $\mathcal{N}=1$ vacua seems to suggest a more profound explanation. It could be for example that the topological model is related to the half–twisting of the $(2, 1)$ model that one would expect to be associated to a $\mathcal{N}=1$ background.

Acknowledgments

We would like to thank Alberto Cattaneo, Marco Gualtieri, Yi Li, Jan Louis, Luca Martucci, Pierre Vanhove, Dan Waldram and Maxim Zabzine for useful discussions.

We also thank the Fields Institute and Perimeter Institute for hospitality during the course of this work, and the organizers and participants of the workshops held there. This work was partially supported by INTAS grant, 03-51-6346, CNRS PICS 2530, RTN contracts MRTN-CT-2004-005104 and MRTN-CT-2004-503369 and by a European Union Excellence Grant, MEXT-CT-2003-509661. MG is supported by European Commission Marie Curie Postdoctoral Fellowship under contract number MEIF-CT-2003-501485. AT is supported by DOE contract DEAC03-76SF00515 and by NSF contract 9870115.

Appendices

A The topological model

Here we explain the appearance of the equation $(d + H\wedge)\Phi = 0$ in the topological theory.

Conventionally, A and B models on a Calabi–Yau are defined as topological twistings of a $(2, 2)$ supersymmetric sigma model. Nevertheless they can be defined directly without any reference to twisting and the resulting geometry turns out to be more general.

An intuitive way of thinking about this generalization is an extension of the usual properties of A and B models on Calabi–Yau manifolds, where they depend only on Kähler and complex structure moduli, respectively. On a manifold of SU(3) structure one may think of A and B models as defined in terms of the pure spinors e^{iJ} or Ω, without a priori imposing both of them to be closed. Given these two examples, it is natural then to expect that one can associate a topological model to any pure spinor.

Let us now see all this more precisely, following [8, 9]. To define a topological model without twisting a $(2, 2)$ model, it is essentially enough to give a space of fields and a BRST differential on it. The space of fields is made of scalars ϕ^m describing the embedding of the worldsheet into the target space, and two spinors $\psi^m_\pm$. It is customary to define $T \oplus T^*$ valued $(\rho^m, \chi_m) = (\frac{1}{2}(\psi^m_+ + \psi^m_-), \frac{1}{2}g_{mn}(\psi^n_+ - \psi^n_-))$.

We can then define a candidate BRST differential by $\{Q, \cdot\}$, where $\{\, ,\}$ is the usual Poisson bracket on the space of fields, and

$$Q = (g_{mn}\partial_0\phi^n, \partial_1\phi^m)(1 + i\mathcal{J})\begin{pmatrix}\rho^m \\ \chi_m\end{pmatrix} .$$

Here 0 and 1 refer to the worldsheet indices, and $\mathcal{J}$ is a map from $T \oplus T^*$ to itself. This is the form proposed in [8] for flat space. The general form has been given in [9] using the formalism of BV superfields[14] (see also [36]); the transformations rules are similar to the ones given for the $(2, 1)$ model in [37]. Imposing that Q defined this way gives an honest differential yields two conditions:
i) $\mathcal{J}^2 = -1$, where 1 the identity in $T \oplus T^*$.

[14] The model can then be written in the usual form by applying the procedure described in [34]. We thank A. Cattaneo and M. Zabzine for discussions on this point.

ii) the i–eigenbundle of $\mathcal{J}$, $L_{\mathcal{J}}$, is closed under the Courant bracket (3.18).
These conditions define a *generalized complex structure* [1,2].

We may now try to translate this into the language of pure spinors. The first condition i) implies that the structure group of $T \oplus T^*$ reduces to U(3,3). As shown in [8], the anomaly cancellation requires a further reduction of the structure group to SU(3,3). This implies that there exists a pure spinor Φ. As mentioned in section 3.3, the integrability of the generalized complex structure, i.e. condition ii) translates into $d\Phi = 0$. Introduction of H–flux (the twisting) modifies this to $(d + H\wedge)\Phi = 0$.

We conclude by putting together this general situation with the particular cases of the A and B models:

model	structure	$Q^2 = 0$	anomaly
A	J_{mn}	$dJ = 0$	—
B	${J_m}^n$	${J_m}^n$ integrable	$c_1 = 0$
generalized	$\mathcal{J}$	$\mathcal{J}$ integrable	$\exists$ pure spinor

B Courant bracket and purely NS $\mathcal{N} = 1$ vacua

In this appendix we briefly discuss the integrability properties of the $\mathcal{N} = 1$ backgrounds with only NS fields (restricted to the case of SU(3) structure) [24,25]. As mentioned in the text, these do not satisfy two pure spinor equations of the form (1.1).

If we denote the space of annihilators of a pure spinor Φ as L_Φ, we may introduce further subspaces of L in the following way. Since the two pure spinors are compatible, $T \oplus T^*$ splits into four subspaces, each annihilating one of the four corners of the Hodge diamond (3.16). Then in an obvious notation, we call these $L_\nearrow$, $L_\nwarrow$, $L_\searrow$, $L_\swarrow$. In models with (2,2) worldsheet supersymmetry ($\mathcal{N} = 2$ spacetime) all four subspaces are closed.

The $\mathcal{N} = 1$ NS backgrounds have pure SU(3) structure and (2,1) worldsheet supersymmetry. The relevant equations are

$$i\partial J = H^{2,1} \ , \qquad d(e^{2\phi} J^2) = 0 \ , \qquad (d + H\wedge)(e^{2\phi}\Omega) = 0. \tag{B.1}$$

where $H^{2,1}$ is the (2,1) component of H. It follows that the manifolds in question have to be complex, and that by scaling the metric, we may define a closed holomorphic three-form. Thus $\Omega = \Phi_-$ is a generalized Calabi–Yau structure, and therefore the space of its annihilators $L_\Omega = L_\nearrow \oplus L_\searrow$ is closed under the twisted Courant bracket (we are using the version of (3.18) with the differential being $d + H\wedge$). On the contrary, e^{iJ} is not a $(d + H\wedge)$–closed and as a consequence, $L_\nearrow \oplus L_\nwarrow$ is not integrable.

However, let us consider $L_\nearrow = L_\Omega \cap L_{e^{iJ}}$. It is generated by elements of the form $\iota_{\partial_i} - g_{i\bar{j}} dz^{\bar{j}}$. An easy computation shows that these are closed under the twisted Courant bracket if and only if the first equation in (B.1) is satisfied.

One may check that $L_\Omega = L_\nearrow \oplus L_\searrow$ and $L_\nearrow = L_\Omega \cap L_{e^{iJ}}$ are the only pieces closed under twisted Courant bracket, thus leaving us with fewer integrable sectors than for $\mathcal{N} = 2$, yet still preserving the generalized Calabi–Yau structure.

References

[1] N. Hitchin, "Generalized Calabi-Yau manifolds," arXiv:math.dg/0209099.

[2] M. Gualtieri, "Generalized complex geometry," Oxford University DPhil thesis, arXiv:math.DG/0401221.

[3] C. Jeschek and F. Witt, "Generalised G(2)-structures and type IIB superstrings," JHEP **0503**, 053 (2005) [arXiv:hep-th/0412280].

[4] F. Witt, "Special metric structures and closed forms", Oxford University DPhil thesis, arXiv:math.DG/0502443.

[5] M.Graña, J.Louis and D.Waldram, "Hitchin functionals in N=2 supergravity", arXiv:hep-th/0505264.

[6] D. Lust and D. Tsimpis, "Supersymmetric AdS(4) compactifications of IIA supergravity," JHEP **0502**, 027 (2005) [arXiv:hep-th/0412250].

[7] M. Graña, R. Minasian, M. Petrini and A. Tomasiello, "Supersymmetric backgrounds from generalized Calabi-Yau manifolds," JHEP **0408**, 046 (2004) [arXiv:hep-th/0406137].

[8] A. Kapustin and Y. Li, "Topological sigma-models with H-flux and twisted generalized complex manifolds," arXiv:hep-th/0407249; A. Kapustin, "Topological strings on noncommutative manifolds," Int. J. Geom. Meth. Mod. Phys. **1**, 49 (2004) [arXiv:hep-th/0310057].

[9] R. Zucchini, "A sigma model field theoretic realization of Hitchin's generalized complex geometry," JHEP **0411**, 045 (2004) [arXiv:hep-th/0409181]; "Generalized complex geometry, generalized branes and the Hitchin sigma model," JHEP **0503**, 022 (2005) [arXiv:hep-th/0501062].

[10] S. J. Gates, C. M. Hull and M. Rocek, "Twisted Multiplets And New Supersymmetric Nonlinear Sigma Models," Nucl. Phys. B **248**, 157 (1984).

[11] S. Fidanza, R. Minasian and A. Tomasiello, "Mirror symmetric SU(3)-structure manifolds with NS fluxes," Commun. Math. Phys. **254**, 401 (2005) [arXiv:hep-th/0311122].

[12] P. Grange and R. Minasian, "Modified pure spinors and mirror symmetry," arXiv:hep-th/0412086.

[13] A. Tomasiello, "Topological mirror symmetry with fluxes," arXiv:hep-th/0502148.

[14] S. Gurrieri, J. Louis, A. Micu and D. Waldram, "Mirror symmetry in generalized Calabi-Yau compactifications," Nucl. Phys. B **654**, 61 (2003) [arXiv:hep-th/0211102].

[15] M. Kontsevich, "Homological algebra of mirror symmetry." ICM Talk, Zürich 1994, alg-geom/9411018.

[16] N. J. Hitchin, "The Geometry of three forms in six-dimensions," J. Diff. Geom. **55**, 547 (2000).

[17] R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa, "Topological M-theory as unification of form theories of gravity," arXiv:hep-th/0411073.

[18] P. A. Grassi and P. Vanhove, "Topological M theory from pure spinor formalism," arXiv:hep-th/0411167.

[19] N. Nekrasov, "A la recherche de la m-theorie perdue. Z theory: Casing m/f theory," arXiv:hep-th/0412021.

[20] V. Pestun and E. Witten, "The Hitchin functionals and the topological B-model at one loop," arXiv:hep-th/0503083.

[21] J. P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, "G-structures and wrapped NS5-branes," Commun. Math. Phys. **247**, 421 (2004) [arXiv:hep-th/0205050].

[22] G. R. Cavalcanti, M. Gualtieri, "Generalized complex structures on nilmanifolds," math.DG/0404451.

[23] A. Lukas and P. M. Saffin, "M-theory compactification, fluxes and AdS(4)," Phys. Rev. D **71**, 046005 (2005) [arXiv:hep-th/0403235].

[24] C. M. Hull, "Superstring Compactifications With Torsion And Space-Time Supersymmetry," Print-86-0251 (CAMBRIDGE)

[25] A. Strominger, "Superstrings With Torsion," Nucl. Phys. B **274** (1986) 253.

[26] M. Graña and J. Polchinski, "Gauge / gravity duals with holomorphic dilaton," Phys. Rev. D **65**, 126005 (2002) [arXiv:hep-th/0106014].

[27] H. B. Lawson and M. –L. Michelsohn, *Spin geometry,* PMS–38, Princeton University Press, 1990.

[28] J. P. Gauntlett, D. Martelli and D. Waldram, "Superstrings with intrinsic torsion," Phys. Rev. D **69**, 086002 (2004) [arXiv:hep-th/0302158].

[29] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest and A. Van Proeyen, "New formulations of D = 10 supersymmetry and D8 - O8 domain walls," Class. Quant. Grav. **18** (2001) 3359 [arXiv:hep-th/0103233].

[30] K. Behrndt and M. Cvetic, "General N = 1 supersymmetric fluxes in massive type IIA string theory," Nucl. Phys. B **708**, 45 (2005) [arXiv:hep-th/0407263].

[31] J. Wess and J. Bagger, "Supersymmetry and Supergravity", Princeton University Press, 1992.

[32] M. Gualtieri, “Generalized geometry and the Hodge decomposition”, [math.DG/04090993].

[33] Y. Kosmann–Schwarzbach, “Derived brackets,” Lett.Math.Phys. 69 (2004) 61-87, [math.DG/0312524].

[34] M. Alexandrov, M. Kontsevich, A. Schwartz and O. Zaboronsky, “The Geometry of the master equation and topological quantum field theory,” Int. J. Mod. Phys. A **12**, 1405 (1997) [arXiv:hep-th/9502010].

[35] G. Dall'Agata, “On supersymmetric solutions of type IIB supergravity with general fluxes,” Nucl. Phys. B **695**, 243 (2004) [arXiv:hep-th/0403220].

[36] M. Zabzine, “Hamiltonian perspective on generalized complex structure,” arXiv:hep-th/0502137.

[37] U. Lindstrom, R. Minasian, A. Tomasiello and M. Zabzine, “Generalized complex manifolds and supersymmetry,” arXiv:hep-th/0405085.